What is Calculus About?

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The goal of this document is to provide you with **intuition** about your first calculus course. We won't be doing any actual math here; we might write down some expressions, but we won't be solving any actual problems (we'll save that for another time).

The storyline: Every math class has (or should have) a storyline; something that ties all of the concepts together and makes it clear why you're learning the things that you are. Here's the story for our class:

This is a class about **limits**. All of the concepts we'll learn in this class (limits themselves, continuity, derivatives, integrals, infinite series, etc.) have something to do with limits. Limits are useful because they give you a way of **zooming in** on functions. This concept (zooming in) might not sound like a big deal, but it shows up all the time in math, physics, engineering, computer science, etc. For example, we might care what happens to a physical system, like a chemical reaction, when we wait a *really long time* (this is zooming in to ∞). Or we might care about what happens when we make something *really small* (zooming in to 0). Right now this might sound kinda weird, but throughout this course you'll have lots of time to begin to appreciate what limits let you do.

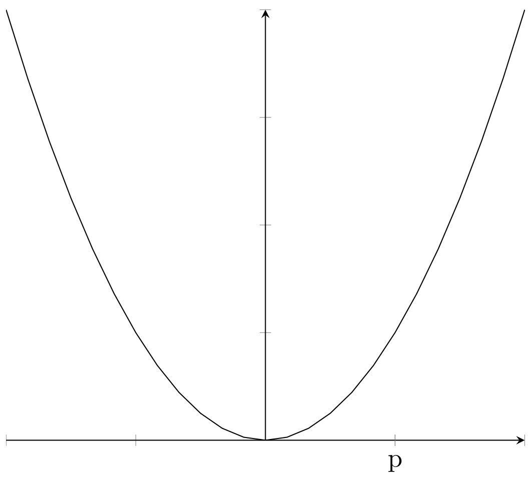
In this class we'll learn about many different things you can do with limits, and different techniques you can use to solve problems involving them.

Now: Let's take a look at this stuff in more detail.

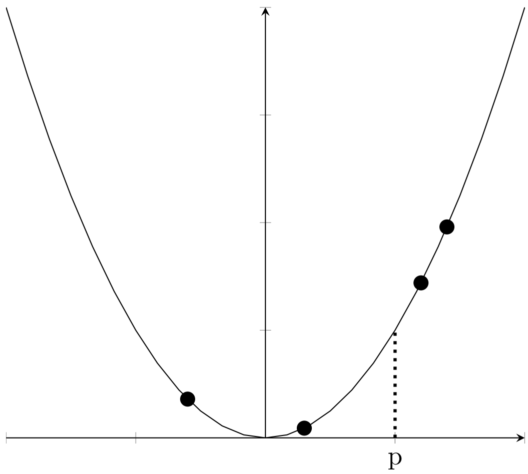
Limits: Zooming In

Intuition: Oftentimes in math, we're given a ton of information. But usually, we don't actually care about all of that info. In that case, we need a way of focusing on the info that we really care about. The concept of **limits** is one way of doing that.

- But what is a limit mathematically? Let's say you have some function, $f(x)$, and a point (**on the x-axis**) that you want to zoom in on (call it p, for *point*). For example:



When you take the limit to the point p, you get closer and closer to the point p, while staying **glued to your function**. It's like you're a bead on a string, moving to the important point:



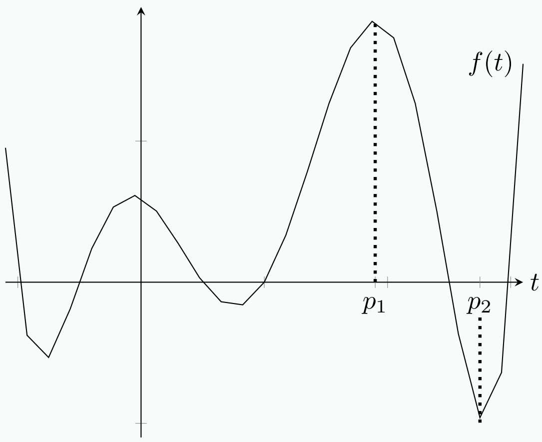
As you can probably tell, you can approach the point p both from the left and from the right. These are called **left and right limits**, and you'll learn about them. But the point is, you approach the point p on the x-axis, while staying glued to your function. This is what the limit does. And we write it like this:

$$\lim_{x \to p} f(x) \tag{1}$$

Let's do some examples to see what the intuition and the mathematical expression $\lim_{x\to p} f(x)$ have to do with each other:

Example 1.1: Too Much Information: Finances

Say we have some function that models the finances of a company over a year. Depending on what we're doing, we might not need all of that info; maybe we just need to know about when the company's finances **crashed** and when they were at their **peak**:



We want to be able to zoom into the two interesting points: p_1 and p_2.

Notation: The limits we are interested in this example are written like this:

$$\lim_{t \to p_1} f(t) \tag{2}$$

$$\lim_{t \to p_2} f(t) \tag{3}$$

How to write down a limit: There are 4 steps to writing down a limit in math-language:

1. Start by writing
$$\lim \tag{4}$$

2. Look at what the dependent variable is. In other words, what is the **label of the horizontal axis**? Whatever the variable is, write it underneath lim, and put a little arrow next to it. In this example, our horizontal axis is called t, so we write
$$\lim_{t \to} \tag{5}$$

3. Decide what point you want to zoom in to. This point **must be on the horizontal axis**. We'll write the point of interest next to our little arrow. In this case we have two points of interest: the first is p_1. So we write
$$\lim_{t \to p_1} \tag{6}$$
The other point of interest is p_2 on the t axis. So in our second limit we wrote $t \to p_2$.

4. Write down the name of the function next to lim. In this case our function is $f(t)$. So our limits are
$$\lim_{t \to p_1} f(t) \tag{7}$$
and
$$\lim_{t \to p_2} f(t) \tag{8}$$

Abstract Thinking: One of the cool things about limits is that you don't just have to use them to zoom in to particular points on a graph; you can also use them more intuitively/abstractly. In any situation where it might make sense to think about zooming in, limits might show up. So for example, if we're running a science experiment and we want to know what happens when we wait a really long time, we can think about **zooming in to infinity**. This is very useful in physics, for example:

Example 1.2: Too Much Information: Physics

Say that we hit a billiard ball with a pool cue, and we want to find out where the ball ends up. If you wait 30 seconds, the ball will probably have settled, and won't move any more. So if you wait one hour, the ball will still be in the same spot. And if you wait two years, the ball will still be in the same spot (we're assuming no one bumps the table or moves the ball etc.). And if you wait a million years, the ball still will be in the same spot.

The point is: In physics, if you have a system that comes to rest after a certain time, then instead of trying to figure out exactly what time things stop moving, you can *take the limit to infinity* (zoom in to infinity), because you know that once the system stops moving, it will be in the same position forever after that.

- Why is this useful?: Oftentimes, limits to infinity are easier than limits to specific numbers.

If we have a function, $B(t)$, that models a ball being hit by a pool cue, then the limit we were talking about above will be written:
$$\lim_{t \to \infty} B(t) \tag{9}$$

$\boxed{\text{Computing limits:}}$ It's nice to be able to write down the notation for the limit, but at the end of the day, we really want to compute the actual value of the limit. So we'll spend a good chunk of time in this class learning how to actually compute limits.

- $\boxed{\text{Limit Techniques:}}$ There are certain techniques that you can use to compute limits faster. These are known as **limit laws**, and we'll learn them as the class goes on.

$\boxed{\text{Continuity:}}$ Some functions have really easy limits. It's nice to know what these functions are, because they're easier to work with. These functions are called **continuous**, and we'll spend some time showing that different functions are continuous. It turns out that most functions you've seen before (e^x, $\sin(x)$, $\cos(x)$, x^2, etc.) are continuous (and that's probably why you've seen them — they're easy to deal with (relatively speaking)).

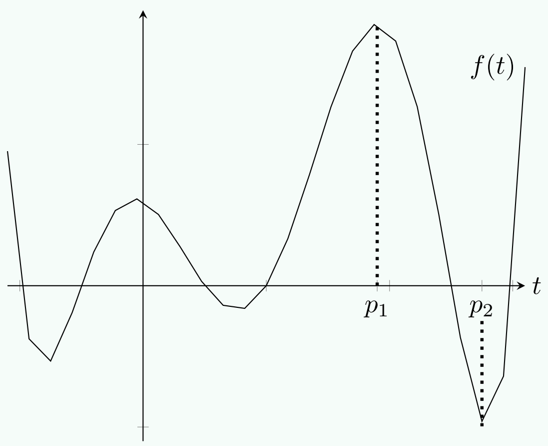
$\boxed{\text{Where should we zoom in?:}}$ Even if we know that we *should* sometimes zoom in on functions, how do we know *where* we should zoom in? How do we know which parts of our function are interesting? There are different approaches to answering this question, but one of the main ones is using **derivatives**.

Derivatives: Shape

$\boxed{\text{Intuition:}}$ Derivatives tell you about the **shape of a function**. This is really useful, because the interesting points of a function usually sit near interesting shapes.

Example 2.1: Interesting Shape: Min and Max

Let's go back to our first example of the function



Do you see how the first point of interest (p_1) sits on a hill of the function? And p_2 sits in a valley. These hills and valleys (in other words, the **maxima** and **minima**) of our function are quite interesting, and we can find them using derivatives, *because* derivatives allow you to study the shape of a function.

$\boxed{\text{Higher Derivatives:}}$ There are different 'levels' of derivatives (1st, 2nd, 3rd, etc.) Each of these tells us something about the shape of our function. In this class we'll only need the first two levels:

- $\boxed{\text{First derivative:}}$ This tells us about whether the function is **increasing or decreasing** at a point. So if you think about being a bead on a string, traveling from left to right, the first derivative tells you about whether you're going up (which is when the first derivative is **positive**) or down (when the first derivative is **negative**), or staying at the same level (when the first derivative is **zero**) at any given moment. In more math-y terminology, the first derivative tells you about the **slope** of the function at a point.

There are different notations for the first derivative. Here are a few:

$$\frac{df}{dx}, \quad f'(x), \quad f^{(1)}(x) \tag{10}$$

- $\boxed{\text{Second derivative:}}$ This tells us about the **curvature** of the function. Basically it tells you if you're a part of a hill (which is when the second derivative is **positive**) or a valley (when the second derivative is **negative**), or neither (when the second derivative is **zero**).

  There are different notations for the second derivative (and all of the other higher derivatives). Here are a few:

$$\frac{d^2 f}{dx^2}, \quad f''(x), \quad f^{(2)}(x) \tag{11}$$

$\boxed{\text{Derivative Techniques:}}$ Just like with limits, we'll learn techniques that make computing derivatives easier (the **derivative rules** and **implicit differentiation**).

$\boxed{\text{Drawing Functions:}}$ Derivatives give you a lot of information about functions. In fact, they give you so much information that you can use them to draw pretty much any function you can write down, even if you've never seen it before. You might already know what the graph of $\sin(x)$ looks like. But what about $x^5 - 3x^4 + \pi x^2 + \frac{35}{2}x - 11$? Or $e^x \sin(x\sqrt{x^3+1}) + \ln(x)$? We'll spend some time in this class figuring out what functions like these look like, using derivatives and limits.

$\boxed{\text{Abstract Thinking:}}$ Just like with limits, you can use derivatives in more intuitive/abstract situations. Derivatives show up in situations involving shape, but also those involving **change**. The reason is, when something changes, we can model it using a function. Then we can study the shape of that function to understand how our thing is changing. For instance, if a flower is growing, we can come up with a function that models its change in height. Then the shape of that function will tell us about how fast the flower is growing. Or if you're walking to the store (changing your position), we can come up with a function that models that. This concept of *change* shows up in a lot of real-world problems, and we'll spend some time in this class solving them. There are two main types of these problems:

- $\boxed{\text{Related Rates:}}$ In these problems, two or more things will be changing at the same time, and we'll have to figure out what happens as a consequence of that. For example, say you want to figure out how fast a rocket is traveling through space. Well, as the rocket uses up fuel, it flies through space: (1) So it's changing its position. But also, as it uses up the fuel, it becomes lighter, since it has less fuel weighing it down: (2) It's changing its mass. And we want to find out (3) how the speed is changing, taking (1) and (2) into consideration.

- $\boxed{\text{Optimization:}}$ In this type of problem, you have a goal (say building a pen to hold in some animals), and there are a bunch of ways to complete that goal (making a square pen, or a circular one, or a triangular one, etc.). But each way of achieving the goal will cost a different amount (maybe a different amount of money, or a different amount of time, etc.) Knowing that, you have to figure out how to achieve the goal with the least amount of cost (spending the least amount of money, or the least amount of time, etc. — whatever the problem asks)

  What does this type of problem have to do with *change*? The idea here is that the cost *changes* depending on how you achieve the goal.

That's enough about derivatives. Our final section of the class has to do with both limits and derivatives, but it might not seem like it at first...

Integrals: Adding Stuff Up

$\boxed{\text{Intuition:}}$ Integrals give you a way of adding stuff up. Now, at first this might sound ridiculous: you know how to add. But I'm not talking about simple addition. Integrals can add **infinitely many things**. But you might be thinking *doesn't that just give you* ∞*? Like* $1 + 1 + 1 + \ldots = \infty$*, right?*.

Yeah, you're right. But think about this: if I want to walk 2 meters, that takes a certain amount of energy. But also so does walking 1 meter. And so does moving my foot the first 1cm, and so does moving my foot 1mm. There are an infinite amount of teeny-tiny distances that I have to move, just to walk 2 meters, and each of those distances costs energy. But it clearly doesn't take me an infinite amount of energy to walk 2 meters. What's happening? How does this work? Integrals give us a way of mathematically dealing with these situations.

Abstract Thinking: This idea of adding up an infinite number of (really small) things shows up all the time. If you're in physics or engineering, you'll be dealing with integrals constantly.

Example 3.1: Integrals: Area

The main use of integrals in this class will be to compute areas. What does this have to do with adding? Think about a rectangle:

We all know the formula for its area (length times width). But we can also compute the area using integrals: we think about every little point inside of the square as having some teeny tiny area, and then we add them all up. That might sound really weird, but we can do it mathematically. And it works! As we'll see later, when you do this weird infinite addition on a rectangle, you end up getting the usual length times width formula.

Example 3.2: Integrals: Physics

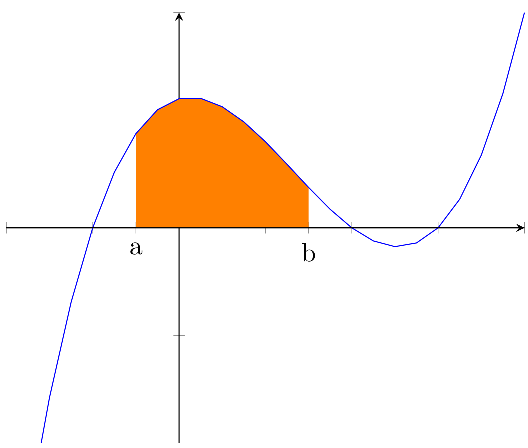
In physics, you often have to deal with situations like particles moving in electric fields, and need to know how much energy is takes for the particle to do that. This sounds like a job for integrals: we look at how much energy it takes the particle to move each teeny tiny distance, and then we add them all up.

Limits and Derivatives: What does this have to do with the topics we've seen so far? Well, as we'll see in class, the way to define integrals involves limits. What about derivatives? Here's the cool thing: one of the most powerful techniques to compute an integral involves derivatives. Basically if you can 'undo a derivative' (find an **antiderivative**) then you can use that to find integrals *way* faster.

I can't really overstate how important this result is (it's actually called the **Fundamental Theorem of Calculus**). Antiderivatives are basically the only way that any human can solve an integral (computers can solve them using computery-techniques, but for people, we pretty much always need antiderivatives).

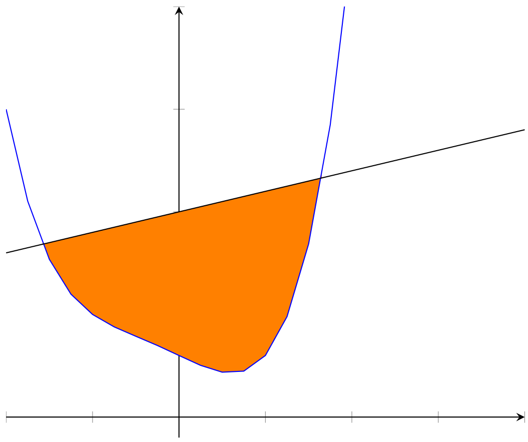
Area: In this class, we'll mainly focus on using integrals to compute areas. There are two types of areas that we'll look at:

- Area under a curve: This is the area between a function and the horizontal axis, like so:



As you can see in the graph above, we don't have to compute all of the area under a given function — we can choose how much we want to do. In other words, the orange region (the area under the curve) only extends from the point a to the point b. We could make it larger or smaller, depending on what we're trying to do.

- $\boxed{\text{Area between functions:}}$ In this situation we look at the areas that sit between two intersecting functions. This is where we get all these weird shapes, like this:



$\boxed{\text{Techniques:}}$ Like with limits and derivatives, there are a number of techniques we'll learn to make solving integrals easier. I already mentioned the main one (the Fundamental Theorem of Calculus), which you'll pretty much always use. But there are others that you'll use in combination with the Fundamental Theorem.

To be honest, integrals are incredibly difficult to solve; even computers have a hard time solving them. There are lots of different techniques available to try to deal with these difficult things. In this class we'll only learn a few (the **substitution rule**, and the Fundamental Theorem), but you'll learn a lot more in Calculus 2.